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On asymptotically correct Timoshenko-like anisotropic beam theory

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Abstract

This paper presents a finite element cross-sectional beam analysis capable of capturing transverse shear effects. The approach uses the variational-asymptotic method and can handle beams of general cross-sectional shape and arbitrary anisotropic material. The work builds on previous works which deal with development of the classical beam theory, which includes only extension, torsion, and bending. A Timoshenko-like formulation is sought to achieve a refined theory with simple boundary conditions. Apart from some simple special cases, it is shown that this problem is overdetermined. However, it is possible to obtain 'the best possible' solution using the least-squares minimization technique. The geometrical meaning of the shear variable for this formulation is also derived. Results are found to be in good agreement with published results for the shear stiffness coefficients for both isotropic and anisotropic beams. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The present cross-sectional beam analysis is developed in order to analyze transverse shear effects in anisotropic prismatic beams of general cross-sectional shape. These effects are important in short beam analysis, high frequency dynamics, and in composite material applications. The paper focuses in the determination of stiffness properties, including shear coefficients, for the cross-sectional analysis of general anisotropic beams when a one-dimensional (1-D) transverse shear measure is taken into consideration.

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There has been a debate over the determination of the shear coefficient spread over the last century. The reader is encouraged to consult Kaneko (1975) for a good survey of the problem. Foeppl (1927), which refers to unpublished notes from 1897, obtained a value of $k = 5/6$ for the shear coefficient for rectangular cross sections using an energy-based approach. Filon (1902) presented his shear theory based on the theory of elasticity and determined the shear coefficient both theoretically and experimentally. Timoshenko (1921, 1922) used Filon's experimental determination of the shear coefficient in his theory on the effect of transverse shear on the transverse vibration of bars. His definition of the shear coefficient was based on the average shear stress and the angle of shear at the neutral axis. Later, Timoshenko (1940) defined the shear correction factor based on geometrical assumptions as the ratio of the average strain over the section to the shear strain at the centroid. This ambiguity in the definition of the shear strain variable points out a deficiency of the geometrically-based methods. A well-known attempt to improve the results was done by Cowper (1966) whose approach gives a more involved expression for the shear correction factor based on the mean angle rotation of the cross section about the neutral axis. His results are closer to those obtained using energy methods, for example, in the case of a rectangular cross section.

Energy-based methods do not need to assume a definition for the shear strain variable. This comes naturally from the expression for the interior shear force resultant as the derivative of the strain energy with respect to the shear variable. This type of approach was used with good results by a number of authors, such as Timoshenko (1958), Dym and Shames (1973), Hoeborn (1993), Renton (1991), and Schramm et al. (1994). It must be mentioned that all the results discussed here assume a relatively slow variation of the shear force along the beam. See also Goodier (1938) and Cowper (1966).

Thin-walled beams are a special case of importance for aeronautical structures. In this case, certain approximations can be made to simplify the problem and render relatively easily obtained approximate solutions for complex configurations; see Bauchau (1985), Rehfield (1985), Bauchau et al. (1987), Stemple and Lee (1988), Chandra et al. (1990), and Rand (1998), all of which treat thin-walled beams made of composite materials. However, since none of these theories is asymptotically correct (nor are they claimed to be), a degree of uncertainty will always exist as to whether or not the results are accurate. Moreover, thin-walled beam theories are really inadequate for analysis of realistic rotor blades and wings, which typically have complex, built-up construction.

For more comprehensive treatment of anisotropic beams, an alternative to making a thin-walled beam approximation is to undertake a generalization of the Saint-Venant solutions, as done by Giavotto et al. (1983). A finite-element-based computer code called *ANBA* (Anisotropic Beam Analysis) or *NABSA* (Nonhomogeneous Anisotropic Beam Section Analysis) was developed by Borri and co-workers which renders a 6×6 cross-sectional stiffness matrix, including the two 1-D transverse shear measures (i.e., in orthogonal directions) along with extension, twist, and bending (the classical 1-D measures). Subsequent treatments by Kosmatka (1992) follow a similar approach (i.e., extended Saint-Venant solution leading to a finite-element-based cross-sectional analysis). Thus, until the present work, only a very few methods exist for determination of the cross-sectional stiffnesses of realistic anisotropic beam cross sections so that the effects of transverse shear are included in the resulting 1-D model. Unfortunately, the accuracy of such models is difficult to assess.

The variational-asymptotic method pioneered by Berdichevsky offers an accurate alternative to exact 3-D solutions when a small parameter is part of the problem. To obtain results that are consistently accurate, asymptotical correctness is the most important thing. By asymptotically correct, we mean that an expansion of the approximate solution in terms of a small parameter agrees with a similar expansion of the exact solution up to a certain order in the small parameter. When this method is applied and carried out to the extent that an asymptotically correct theory is obtained, this result is the most accurate theory possible for a given degree of complexity. This method has been applied for the transverse shear problem of prismatic isotropic beams by Berdichevsky and Kvashnina (1976) and to isotropic beams having initial twist and curvature by Berdichevsky and Starosel'skii (1983). Although it has been applied to the analysis of initially twisted and curved anisotropic beams by, for example, Cesnik (1994), Cesnik, Hodges and Sutyrin (1996a), and Cesnik and Hodges (1997), it has not been applied toward the development of a refined theory which treats transverse shear deformation of composite beams. Although Cesnik (1994) did develop a so-called 'alternative theory' to treat transverse shear effects, this theory is not asymptotically correct nor was it claimed to be.

Therefore, in this paper a refined theory to treat shear deformation in composite beams is carried out via the variational-asymptotic method. The approach and the results are somewhat similar to those of a similar analysis by Sutyrin and Hodges (1996) for laminated composite plates. In that work an asymptotically correct refined theory having the form of a Reissner-like plate theory (analogous to a Timoshenko-like beam theory) does not always exist for plates made of anisotropic materials. However, by means of certain optimization procedures a Reissner-like theory can be obtained that is quite close to being asymptotically correct.

2. Refined theory formulation

The classical theory of anisotropic beams, as developed by Hodges et al. (1992) and by Cesnik and Hodges (1997), contains only four generalized strain measures: extension, torsion, and bending in two directions. The different levels of accuracy are assessed based on asymptotic series in terms of several small parameters. One such parameter is the maximum strain in the beam, ϵ. Classical theory assumes $\epsilon \ll 1$. Another small parameter for beams is h/l, where h is a characteristic dimension of the cross section, and l is the wavelength of the deformation along the beam. The approximation made is simply that $h \ll l$, which naturally leads to the neglect of derivatives of the 1-D strain measures with respect to the axial coordinate. Transverse shear effects, on the other hand, stem from refinements to the classical beam analysis which lead to keeping terms in the energy up through h^2 relative to l^2. This has the effect of allowing some of these derivatives to remain in the theory. It should be noted that the only approximations in the present work stem from the order up to which small parameters in the strain energy are retained, the finite element solution of the governing equations, and the least squares solutions that are invoked when the resulting system of algebraic equations is overdetermined.

2.1. Beam kinematics

Normal cross sections of the beam and the Jaumann strain definition are considered. The coordinate systems are shown in Fig. 1. The position vector of an arbitrary point on the cross section is given by

$$\bar{\mathbf{r}}(x_1,x_2,x_3) = \mathbf{r}(x_1) + x_\alpha \hat{\mathbf{t}}_\alpha(x_1) \tag{1}$$

in the undeformed state and

$$\bar{\mathbf{R}}(x_1,x_2,x_3) = \mathbf{R}(x_1) + x_\alpha \hat{\mathbf{T}}_\alpha(x_1) + w_i(x_1,x_2,x_3)\hat{\mathbf{T}}_i(x_1) \tag{2}$$

in the deformed state, where w is the warping field and $\hat{\mathbf{t}}_i$ and $\hat{\mathbf{T}}_i$ are orthogonal triads of base vectors for coordinate systems associated with the undeformed and deformed beam, respectively. The unit vector $\hat{\mathbf{t}}_1 = \partial\bar{\mathbf{r}}/\partial x_1$ is tangent to the undeformed beam coordinate axis along which x_1 is defined. The unit vector $\hat{\mathbf{T}}_1$ is tangent to the reference line of the deformed beam. The unit vectors $\hat{\mathbf{t}}_\alpha$ are parallel to the reference cross-sectional plane of the undeformed beam. (Note that Greek indices range from 2 to 3, while Latin indices can have values 1, 2, or 3).

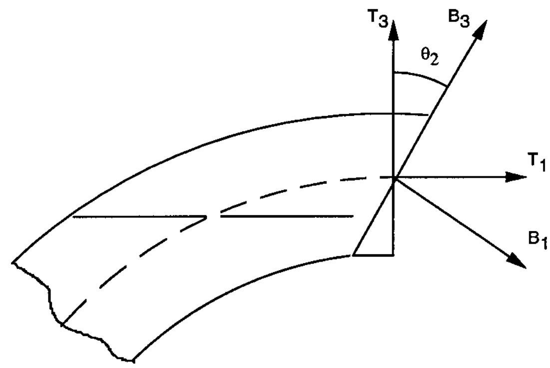

Fig. 1. Coordinate systems used for transverse shear formulation.

Another triad which must be considered is $\hat{\mathbf{B}}_i$. The unit vectors $\hat{\mathbf{B}}_\alpha$ are fixed in a frame associated with the deformed beam reference cross-sectional plane. This frame is one in which the material points of the undeformed beam cross-sectional plane undergo small deformation as pointed out by Cesnik and Hodges (1997). Thus, $\hat{\mathbf{B}}_1$ deviates from the tangent position, but the triad $\hat{\mathbf{B}}_i$ still forms an orthogonal reference system. The orientation angles which relate $\hat{\mathbf{T}}_i$ and $\hat{\mathbf{B}}_i$ are small but finite quantities.

2.2. The strain field

Using the method presented by Danielson and Hodges (1987) for small strain and small local rotation, the strain field for prismatic beams is obtained

$$\Gamma = \left\{\Gamma^*_{11} 2\Gamma^*_{12} 2\Gamma^*_{13} \Gamma^*_{22} 2\Gamma^*_{23} \Gamma^*_{33}\right\}^T = \Gamma_\varepsilon \varepsilon + \Gamma_h w + \Gamma_l w', \tag{3}$$

where the warping field is

$$w = \{w_1 \; w_2 \; w_3\}^T \tag{4}$$

and the nonlinear generalized 1-D strain measures are

$$\varepsilon = \{\gamma_{11} \; \beta_1 \; \beta_2 \; \beta_3\}^T. \tag{5}$$

Here $()'$ is the partial derivative with respect to x_1, γ_{11} is the stretch, β_1 is the twist per unit length, and β_α is the bending curvature — all of which can be written in geometrically exact form for the triad $\hat{\mathbf{T}}_i$; see the section on the specialized equations for zero transverse shear strain in Hodges (1990). The operator matrices in Eq. (3) are defined as:

$$\Gamma_h = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix}, \tag{6}$$

$$\Gamma_\varepsilon = \begin{bmatrix} 1 & 0 & x_3 & -x_2 \\ 0 & -x_3 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

and

$$\Gamma_l = \begin{bmatrix} I_3 \\ O_3 \end{bmatrix}, \tag{8}$$

where I_3 is the 3×3 identity matrix, and O_3 is the 3×3 zero matrix.

Here, all the quantities of the deformed geometry are still referred to the tangential system $\hat{\mathbf{T}}_i$ and are all dimensional. The warping field must satisfy four constraint equations which serve to remove the rigid-body modes from the warping field. These equations are not unique; we follow Cesnik and Hodges (1997) and write them as

$$\langle w_i \rangle = 0$$

$$\langle x_2 w_3 - x_3 w_2 \rangle = 0, \tag{9}$$

and enforce them in their discretized sense and weakly (i.e., with Lagrange multipliers — see below). The angle-bracket operator, used throughout the paper, is defined as

$$\langle \cdot \rangle = \frac{1}{A}\int_A \cdot \, \mathrm{d}x_2 \, \mathrm{d}x_3, \tag{10}$$

where A is the cross-sectional area.

2.3. First approximation (classical theory)

The strain energy is given by:

$$2U = \left\langle \Gamma^T D \Gamma \right\rangle \tag{11}$$

The first approximation in determining the warping field (see Cesnik et al. (1996a, 1996b)) comes from keeping only terms up to order $\mu\epsilon^2$ in the strain energy, where μ is the value of a typical material modulus and ϵ is the maximum magnitude of the strain. The first approximation of the warping is discretized using shape functions N so that $w(x_1,x_2,x_3)=N(x_2,x_3)V_0(x_1)$ where the shape functions N have the order of unity. The strain energy is therefore written approximately as $U=U_0$ so that

$$2U_0 = V_0^T E V_0 + 2V_0^T D_{h\varepsilon}\epsilon + \epsilon^T D_{\varepsilon\varepsilon}\epsilon, \tag{12}$$

where the following definitions are introduced

$$E = \langle [\Gamma_h N]^T D [\Gamma_h N] \rangle$$

$$D_{\varepsilon\varepsilon} = \langle [\Gamma_\varepsilon]^T D [\Gamma_\varepsilon] \rangle$$

$$D_{h\varepsilon} = \langle [\Gamma_h N]^T D [\Gamma_\varepsilon] \rangle \tag{13}$$

Minimizing the energy with respect to the warping V_0 and eliminating the rigid body modes by introducing the Lagrange multipliers λ, one obtains

$$D_{h\varepsilon}\varepsilon + EV_0 = H\psi_{\text{cl}}\lambda, \tag{14}$$

where

$$H = \langle N^T N \rangle \tag{15}$$

The discretized condition to have the rigid-body modes removed from the warping field is

$$V_0^T H \psi_{\text{cl}} = 0, \tag{16}$$

where the rigid-body modes are obtained from the null space of matrix E

$$E\psi_{\text{cl}} = 0. \tag{17}$$

Here, ψ_{cl} is the discrete version of the left hand sides of Eq. (9). Normalizing ψ_{cl} such that

$$\psi_{\text{cl}}^T H \psi_{\text{cl}} = I_4. \tag{18}$$

Cesnik et al. (1996a, 1996b) showed that the Lagrange multipliers are

$$\lambda = \psi_{\text{cl}}^T D_{h\varepsilon}\varepsilon. \tag{19}$$

Defining a symmetric matrix E_{cl}^+ such that

$$E_{\text{cl}}^+ E = I - \psi_{\text{cl}}\psi_{\text{cl}}^T H, \tag{20}$$

one can solve the equation for the discretized warping V

$$V = -E_{\text{cl}}^+ D_{h\varepsilon}\epsilon \triangleq \hat{V}\epsilon. \tag{21}$$

One can now write the strain energy function for classical beam theory as

$$2U_0 = \epsilon^T A \epsilon, \tag{22}$$

where

$$A = D_{\varepsilon\varepsilon} + \hat{V}_0^T E \hat{V}_0 + +2D_{h\varepsilon}^T \hat{V}_0, \tag{23}$$

which is valid through order $\mu\epsilon^2 h^0/l^0 = \mu\epsilon^2$, which means that terms of order h/l are neglected compared to unity.

2.4. Next approximation

According to the variational-asymptotic method, one must now consider the perturbation in the warping field

$$w = \underbrace{w_{(0)}}_{\epsilon} + \underbrace{w_{(1)}}_{\epsilon\frac{h}{l}}. \tag{24}$$

Recall that the finite element discretization of the warping is given by $w = NV$ where N represents the matrix of the shape functions. For the above representation, the same shape functions for the perturbation are used, viz.,

$$w_{(0)} = NV_0, \quad w_{(1)} = NV_1 \tag{25}$$

and

$$V = V_0 + V_1. \tag{26}$$

Similar to the treatment of the classical warping, V_1 does not contribute to the rigid motion of the cross section so that

$$V_1^T H \psi_{\mathrm{cl}} = 0. \tag{27}$$

Use of Eq. (26) permits us to rewrite Eq. (3) in the form

$$\Gamma = \underbrace{\Gamma_\varepsilon \varepsilon + \Gamma_h N V_0}_{\epsilon} + \underbrace{\Gamma_h N V_1 + \Gamma_l N V_0{}'}_{\epsilon\frac{h}{l}} + \underbrace{\Gamma_l N V_1{}'}_{\epsilon\left(\frac{h}{l}\right)^2}. \tag{28}$$

Note that orders of magnitude in Eq. (28) are not assumed, but rather uniquely defined by substitution of Eq. (28) into Eq. (11). Indeed, orders of terms with ε and V_0 are known from the previous approximation, while minimization of Eq. (11) which is quadratic with respect to the unknown V_1 requires the leading quadratic and linear terms with respect to this unknown to be of the same order. Note that the following terms vanish due to the Euler–Lagrange equation for V_0 (which directly follows from Eqs. (14) and (27))

$$2(\Gamma_\varepsilon \varepsilon + \Gamma_h N V_0)^T D \Gamma_h N V_1 = 0. \tag{29}$$

Substituting Eq. (28) into Eq. (11) allows one to write consistently all terms in the energy with respect to h/l in the form

$$2U = 2\underbrace{U_0}_{\epsilon} + 2\underbrace{U_1}_{\epsilon\frac{h}{l}} + 2\underbrace{U_2}_{\epsilon\left(\frac{h}{l}\right)^2}. \tag{30}$$

Terms of order U_0 will correspond to the classical approximation (22). Terms of order $(h/l)^1$ do not contain V_1 due to Eq. (29) and can be written as

$$2U_1 = 2\left(V_0^T D_{hl} + \varepsilon^T D_{\varepsilon l}\right) V_0{}'. \tag{31}$$

Here and below, the following notation is used

$$D_{hl} = \left\langle [\Gamma_h N]^T D [\Gamma_l N] \right\rangle$$

$$D_{\varepsilon l} = \left\langle [\Gamma_\varepsilon N]^T D [\Gamma_l N] \right\rangle$$

$$D_{ll} = \left\langle [\Gamma_l N]^T D [\Gamma_l N] \right\rangle \tag{32}$$

Terms of order $(h/l)^2$ can be expressed as

$$2U_2 = \varepsilon'^T \hat{V}_0^T D_{ll} \hat{V}_0 \varepsilon' + 2\mathscr{L}(V_1) + \mathscr{Q}(V_1, V_1), \tag{33}$$

where the linear $\mathscr{L}$ and quadratic $\mathscr{Q}$ operators (with respect to V_1) are

$$2\mathscr{L}(V_1) = 2V_1^T D_{hl} \hat{V}_0 \varepsilon' + 2\varepsilon^T \left(D_{\varepsilon l} + \hat{V}_0 D_{hl}\right) V_1{}' \tag{34}$$

$$\mathscr{Q}(V_1, V_1) = V_1^T E V_1. \tag{35}$$

Integration by parts and grouping Eq. (34) yields

$$2\mathscr{L}(V_1) = -2V_1^T D_{\varepsilon l}^T \varepsilon' + 2V_1^T \left(D_{hl} - D_{hl}^T\right) \hat{V}_0 \varepsilon'. \tag{36}$$

This leads to the following Euler–Lagrange equation:

$$EV_1 = \left[D_{\varepsilon l}^T + \left(D_{hl}^T - D_{hl}\right)\hat{V}_0\right]\varepsilon' + H\psi_{\mathrm{cl}}\lambda_1, \tag{37}$$

where λ_1 is a Lagrange multiplier which enforces the constraint of Eq. (27). Using a procedure identical to one that was used previously to find λ, λ_1 is obtained and substituted into Eq. (37). This leads to an expression for the second warping approximation

$$EV_1 = \left(I - H\psi_{\mathrm{cl}}\psi_{\mathrm{cl}}^T\right)\left[D_{\varepsilon l}^T + \left(D_{hl}^T - D_{hl}\right)\hat{V}_0\right]\varepsilon'. \tag{38}$$

This system must be solved for $\hat{V}_1$ where $V_1 = \hat{V}_1\varepsilon'$ was considered. This is accomplished similarly to before by introducing the matrix E_{cl}^+, such that

$$V_1 = E_{\mathrm{cl}}^+ D_s \varepsilon' \triangleq \hat{V}_1 \varepsilon', \tag{39}$$

where a new matrix was defined

$$D_s = D_{\varepsilon l}^T + \left(D_{hl}^T - D_{hl}\right)\hat{V}_0. \tag{40}$$

Once $\hat{V}_1$ is found, it can be substituted back into Eqs. (34) and (35), which leads to determination of all the terms in Eq. (33). Thus, the energy which is valid up through the second order can be expressed as

$$2U = \varepsilon^T A\varepsilon + 2\varepsilon^T B\varepsilon' + \varepsilon'^T C\varepsilon' + 2\varepsilon^T D\varepsilon'', \tag{41}$$

where A is defined above and

$$B = \left(\hat{V}_0^T D_{hl} + \varepsilon D_{\varepsilon l}\right)\hat{V}_0,$$

$$C = \hat{V}_0^T D_{ll}\hat{V}_0 + 2\hat{V}_1^T D_{hl}\hat{V}_0 + \hat{V}_1 E\hat{V}_1$$

$$D = D_{\varepsilon l} + \hat{V}_0 D_{hl}. \tag{42}$$

3. Transformation to Timoshenko-like form

The Timoshenko-like formulation that is sought here implies writing the strain energy in the form

$$2U = \bar{\varepsilon}^T S\bar{\varepsilon}, \tag{43}$$

where S is the corresponding 6×6 stiffness matrix and $\bar{\varepsilon}$ is an extended column matrix of 1-D strain measures,

$$\bar{\varepsilon} = \{\gamma_{11}\ \kappa_1\ \kappa_2\ \kappa_3\ 2\gamma_{12}\ 2\gamma_{13}\}^T \tag{44}$$

and where the bending curvature in the relation between the curvatures in the tangential system β_i and the curvatures in the cross-sectional system κ_i can be shown to be given by

$$\kappa_1 = \beta_1$$

$$\kappa_\alpha = \beta_\alpha + e_{\alpha\beta}2\gamma_{1\beta}'. \tag{45}$$

Here, $e_{\alpha\beta}$ is the permutation symbol, and terms of higher order have been neglected.

Consistent with the asymptotic analysis considered so far, it can be shown that the linear 1-D equilibrium relation can be used without any loss of accuracy. It also follows that the shear force varies along the beam with characteristic length l. (For the isotropic case the results are different for a more rapidly varying shear force, as discussed by Renton (1991). However, this is a case beyond the Timoshenko-like modeling of the beam and will not be treated here). Thus, the equilibrium equations reduce to the following. First, the force equilibrium equations are

$$F_i' = 0 \tag{46}$$

and the moment equilibrium equations are

$$M_1' = 0,$$

$$M_2' = F_3$$

$$M_3{}' = -F_2. \tag{47}$$

Expressions for the resultant interior forces are

$$F_1 = \frac{\partial U}{\partial \gamma_{11}},$$

$$F_2 = \frac{\partial U}{\partial (2\gamma_{12})},$$

$$F_3 = \frac{\partial U}{\partial (2\gamma_{13})},$$

$$M_1 = \frac{\partial U}{\partial \kappa_1},$$

$$M_2 = \frac{\partial U}{\partial \kappa_2}$$

$$M_3 = \frac{\partial U}{\partial \kappa_3}. \tag{48}$$

Following procedures conducted for isotropic beams by Berdichevsky and Kvashnina (1976) and Berdichevsky and Starosel'skii (1983), we effectively redefine the 1-D variables by setting the last term in Eq. (41) to zero. This leads to

$$2U = \varepsilon^T A \varepsilon + 2\varepsilon^T B \varepsilon' + \varepsilon'^T C \varepsilon'. \tag{49}$$

We can now incorporate Eq. (45) into a relation between classical 1-D strain measures (i.e., Euler–Bernoulli) and Timoshenko-like 1-D strain measures such that

$$\varepsilon = \varepsilon_t + D\gamma', \tag{50}$$

where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{51}$$

and

$$\gamma = \begin{Bmatrix} 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix}. \tag{52}$$

Expressing the strain energy in terms of the Timoshenko beam strain measures gives

$$2U = \left(\varepsilon_t + D\gamma'\right)^T A \left(\varepsilon_t + D\gamma'\right) + 2\varepsilon_t^T B \varepsilon_t{}' + \varepsilon_t{}'^T C \varepsilon_t{}'. \tag{53}$$

Neglecting higher order terms, expanding the expression, and dropping the index 't',

$$2U = \varepsilon^T A\varepsilon + 2\varepsilon^T AD\gamma' + 2\varepsilon^T B\varepsilon' + \varepsilon' C\varepsilon'. \tag{54}$$

According to the Timoshenko-like form given by Eq. (43), the energy can also be written as

$$2U = \varepsilon^T X\varepsilon + 2\varepsilon^T F\gamma + \gamma^T G\gamma, \tag{55}$$

which implies that S is of the form

$$S = \begin{bmatrix} X & F \\ F^T & G \end{bmatrix}. \tag{56}$$

Matrices X, F and G are the unknowns of the problem. Next, let us add and subtract some terms in Eq. (54) and conveniently rearrange the expression;

$$2U = \varepsilon^T A\varepsilon + \varepsilon^T F\gamma + \gamma^T F^T\varepsilon + \gamma^T G\gamma + \left[2\varepsilon^T AD\gamma' + 2\varepsilon^T B\varepsilon' + \varepsilon'^T C\varepsilon' - \varepsilon^T F\gamma - \gamma^T F^T\varepsilon - \gamma^T G\gamma\right]. \tag{57}$$

To have the desired form for the energy, the expression in brackets must be identically zero. Denote this expression with R. The variables ε and ε' are considered as independent variables for the cross-sectional problem. To express γ as a function of ε and ε', the equilibrium equations Eq. (47) are used. Thus,

$$e_{\alpha\beta} M_\alpha = F_\beta. \tag{58}$$

Defining the column matrix of resultant forces and moments as

$$\mathscr{F} = \{F_1 \ M_1 \ M_2 \ M_3\}^T \tag{59}$$

and using Eqs. (48) and (49), one obtains

$$\mathscr{F} = A\varepsilon + B\varepsilon' \tag{60}$$

or, taking the derivative,

$$\mathscr{F}' = A\varepsilon' \tag{61}$$

from the equilibrium equations it can be shown that $\varepsilon'' \approx 0$. Using the D matrix as defined in Eq. (51), one can write

$$\begin{Bmatrix} -M_3 \\ M_2 \end{Bmatrix} = -D^T\mathscr{F}. \tag{62}$$

To obtain the expression for the shear forces, Eq. (55) together with Eq. (48) must be used. By taking the derivatives with respect to γ, one gets

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = G\gamma + F^T\varepsilon. \tag{63}$$

Replacing the corresponding relations in Eq. (47),

$$-D^T A\varepsilon' = G\gamma + F^T\varepsilon, \tag{64}$$

which finally gives the expression for γ:

$$\gamma = -G^{-1}F^T\varepsilon - G^{-1}D^T A\varepsilon' \tag{65}$$

or, denoting $A^* = -D^T A$,

$$\gamma = G^{-1} A^* \varepsilon' - G^{-1} F^T \varepsilon \tag{66}$$

and, by differentiating,

$$\gamma' = -G^{-1} F^T \varepsilon'. \tag{67}$$

Replacing the last two expressions in R and doing all the simplifications, one is left with

$$R = 2\varepsilon^T (A^{*T} G^{-1} F^T + B)\varepsilon' + \varepsilon'^T (C - A^{*T} G^{-1} A^*)\varepsilon' + \varepsilon^T (F G^{-1} F^T)\varepsilon. \tag{68}$$

Since each of the individual coefficients of the bilinear form has to vanish, the second term gives

$$C = A^{*T} G^{-1} A^*. \tag{69}$$

Matrix G is the unknown and, since it is symmetric, it contains 3 scalar unknown variables. Matrix C is 4×4 and, since it is also symmetric, there are 10 distinct equations. This leads to an overdetermined problem. Since A^* is not a square matrix, the pseudoinverse technique can be used to obtain the solution in the sense of a least-squares minimization problem, so that

$$G = \hat{A} (A^* C A^{*T})^{-1} \hat{A}, \tag{70}$$

where $\hat{A} = A^* A^{*T}$. From the first term in Eq. (68), we note that

$$B = -A^{*T} G^{-1} F^T. \tag{71}$$

An equation for F is now readily obtained by pre-multiplying by A^*, then by $\hat{A}^{-1}$, and finally by G, so that

$$F = -B^T A^{*T} \hat{A}^{-1} G. \tag{72}$$

One can notice that determining F was also an overdetermined problem since the matrix equation Eq. (71) consists of 16 scalar equations, because F and B in general are not symmetric. The 8 elements of F are the scalar unknowns.

The last term in Eq. (68) cannot be made zero. However, at a closer look, this term does not contradict the assumed form for the energy Eq. (55) if X is taken as

$$X = A + F G^{-1} F^T. \tag{73}$$

Thus, all the elements of the 6×6 stiffness matrix S have been found. It is important to acknowledge that this is not an exact process, but actually it involves a minimization problem given by the existence of a total of 26 equations and only 11 unknowns. It is remarkable that, even though the two approaches are essentially different, Sutyrin and Hodges (1996) arrived at a similar conclusion for the plate analysis when 78 equations and 33 unknowns are found, exactly three times the numbers in the present approach. There are cases when some of the equations are identities and thus, it is possible to have situations when the solution is exact. One such example is given by isotropic material and symmetric cross sections with the reference axes taken as the symmetry axes. However, in accordance with the rigor of the above development, we expect that the solution for the general case is sufficient for most applications requiring a Timoshenko-like model.

4. The meaning of the shear strain variable

Since there are two main approaches to the transverse shear formulation: the geometric approach and the energy approach, it makes sense to make a connection between them by extracting the geometrical definition of the shear strain variable. In the present formulation, the definition of the shear strain variable $2\gamma_{12}$ comes from the relation for the resultant shear force:

$$F_2 = \frac{\partial U}{\partial(2\gamma_{12})}. \tag{74}$$

Let us assume, for simplicity, a homogeneous, isotropic cross section. The resultant of the shear stresses can be written

$$F_2 = \int_A \sigma_{12}\,\mathrm{d}A = \int_A G2\Gamma_{12}\,\mathrm{d}A = GA\langle 2\Gamma_{12}\rangle \tag{75}$$

Using Eq. (3), the expression for the strain becomes

$$2\Gamma_{12} = -x_3\kappa_1 + \frac{\partial\left(w_1^{(0)} + w_1^{(1)}\right)}{\partial x_2} + \left\langle w_2^{(0)}\right\rangle'. \tag{76}$$

Taking $\langle 2\Gamma_{12}\rangle$ and assuming that the reference point is at the centroid, the first term in Eq. (76) is zero. Because $\langle w_2^{(0)}\rangle = 0$, the last term also turns out to be zero. Thus, we have

$$\langle 2\Gamma_{12}\rangle = \left\langle \frac{\partial\left(w_1^{(0)} + w_1^{(1)}\right)}{\partial x_2}\right\rangle = \left\langle\frac{\partial w_1}{\partial x_2}\right\rangle \tag{77}$$

Thus, the shear force becomes

$$F_2 = GA\left\langle\frac{\partial w_1}{\partial x_2}\right\rangle \tag{78}$$

For the case of isotropic material, it is customary to define the shear coefficient k based on the relation

$$F_2 = kGA2\gamma_{12}. \tag{79}$$

For isotropic, symmetrical cross sections, it can be shown that the shear rigidity is given by

$$S_{22} = \frac{A_{44}^2}{C_{44}} = kGA. \tag{80}$$

This provides an expression for the shear coefficient k. From Eqs. (78) and (79), the expression for $2\gamma_{12}$ becomes

$$2\gamma_{12} = \frac{1}{k}\left\langle\frac{\partial w_1^{(1)}}{\partial x_2}\right\rangle, \tag{81}$$

where the fact that, for this particular case, $w_1^{(0)} = 0$ was used. In Eq. (80), A_{44} is a function only of $w^{(0)}$, while C_{44} is a more complex function of both $w^{(0)}$, $w^{(1)}$ and their derivatives so that $2\gamma_{12}$ does not

Table 1
Shear coefficients for an isotropic beam with square cross section

Source	k
Roark (1954)	0.833
Berdichevsky and Starosel'skii (1983)	$5/6=0.833$
Present ($\nu=0.3$)	0.834

have a simple expression. However, Eq. (81) shows that $2\gamma_{12}$ depends only on the average of the cross-sectional rotation about x_3 as given by $\langle \frac{\partial w_1^{(1)}}{\partial x_2} \rangle$.

For the general case of anisotropic and/or nonsymmetric cross sections, the expression will still depend on the average cross-sectional rotation, which will be given by both warping components in this case, and the shear coefficient will have a more complicated form.

5. Numerical results

The computer program *VABS* of Cesnik and Hodges (1997) has been upgraded to include the above analysis. In this section, numerical results from *VABS* for both isotropic and anisotropic cases are presented and compared with available published results.

5.1. Isotropic material

In general, results based on energy methods tend to be more consistent compared with the results obtained by geometric method and, since the method presented here is an energy based method, only such results from the literature are used for comparison.

For isotropic materials and symmetric cross sections with the reference axes coinciding with the symmetry axes, the shear rigidity is based on the shear coefficient k defined by the relation: $S=kGA$. Tables 1 and 2 show the comparison between present results and other results from the literature for the case of a rectangular cross section and a circular tube, respectively. It is interesting that several researchers (e.g., those quoted by Mason and Herrmann, 1968; Kaneko, 1975; and Renton, 1991) present results which, only when $\nu=0$, coincide with those presented in Tables 1 and 2. It would be tempting to conclude that Berdichevsky and Starosel'skii (1983) ignore Poisson's ratio, but they do not. Furthermore, results given in Berdichevsky and Kvashnina (1976) seem to vary with Poisson's ratio in cases for which those of Berdichevsky and Starosel'skii (1983) do not. The reasons for this discrepancy are not known to the authors; they may stem from a change of variable used in the former but not in the latter.

A special case for isotropic material is given by nonsymmetric cross sections or cross sections for which the reference axes are not the principal bending axes. This generalization was first attempted by

Table 2
Shear coefficients for an isotropic circular tube (m = int diam/ext diam = 0.92)

Source	k
Berdichevsky and Starosel'skii (1983)	$\frac{6(1+m^2)^2}{7+34m^2+7m^4}=0.5014$
Present ($\nu=0.3$)	0.5014

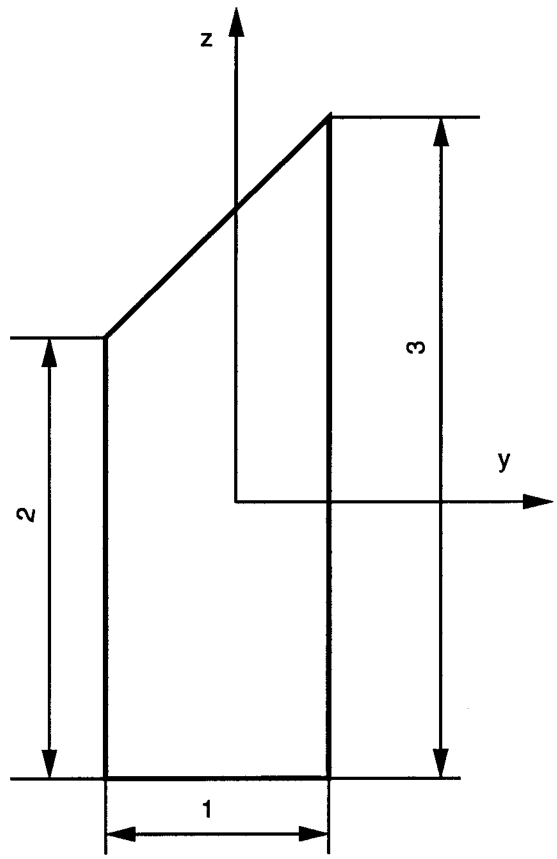

Fig. 2. Trapezoidal cross section.

Mason and Herrmann (1968). In this case, cross-terms usually appear in the shear rigidities. Such terms appear for symmetric cross sections, if the reference axes are not the principal bending axes. For nonsymmetric cross sections the principal shear axes can be defined. They are, in general, different from the principal bending axes. Such an example is given by the trapezoidal section in Fig. 2. Results are compared with the values obtained by Schramm et al. (1994). The shear deformation coefficients, α_{22}, α_{33}, α_{23} and α_{32}, are defined based on the shear flexibilities Eq. (82):

$$2\gamma_{12} = \alpha_{22}\frac{F_2}{GA} + \alpha_{23}\frac{F_3}{GA}$$

$$2\gamma_{13} = \alpha_{32}\frac{F_3}{GA} + \alpha_{33}\frac{F_2}{GA}. \tag{82}$$

The variation of the shear coefficients with the Poisson's ratio is presented in Table 3 for comparison with the results of Schramm et al. (1994); very good agreement can be seen. It can be shown that, even if the material is isotropic, the trapezoidal section is not an exact case in the sense defined earlier.

Another interesting case for isotropic beams is the rectangular cross section. In the literature prior to Renton (1991), there is no differentiation with respect to which of the principal axes the shear coefficient

Table 3
Shear coefficients for an isotropic beam with trapezoidal cross section

Schramm et al. (1994)		$\nu=0$	$\nu=0.3$	$\nu=0.4$	$\nu=0.5$
	α_{22}	1.3134	1.4654	1.5480	1.6305
	α_{23}	−0.0638	−0.0760	−0.0827	−0.0894
	α_{33}	1.1695	1.1707	1.1714	1.1720
Present	α_{22}	1.3132	1.4682	1.5509	—
	α_{23}	−0.0638	−0.0765	−0.0833	—
	α_{33}	1.1695	1.1707	1.1716	—

Table 4
Values for C_2

b/a	C_2
1.0	0.1392
2.0	0.3511
4.0	0.5900
10.0	0.8229

applies. Most of the energy derivations give the value $k = 5/6$ for the shear coefficient in either principal direction. Renton considered the dependency of the results on the aspect ratio of the cross section a/b, where b is the breadth and a being the depth of the cross section. He proposes the following relation for $b/a \geq 1.0$

$$k = 1.2 + C_2\left(\frac{\nu}{1+\nu}\right)^2\left(\frac{b}{a}\right)^2 \tag{83}$$

with C_2, for $\nu=0.3$, taking the values in Table 4. It is noticeable that neither Berdichevsky, who also uses the variational-asymptotic method to determine the shear coefficients, nor Washizu capture the variation of the shear coefficient with the Poisson ratio or the aspect ratio of the rectangular cross section. Table 5 presents the values obtained using Eq. (83) together with the results of the present approach for a square cross section ($b/a = 1$). Table 6 shows the values from Renton's formula and the results obtained with the present approach relative to the softer direction when the ratio of the cross section is varied and $\nu=0.3$.

Table 5
Variation of shear coefficient with Poisson ratio

ν	0.0	0.3	0.4
$1/k$ (Renton, 1991)	1.200	1.207	1.211
$1/k$ (present)	1.199	1.207	1.211

Table 6
Variation of shear coefficient with the breadth-to-depth ratio relative to the soft direction

b/a	1	2	4	10
$1/k$ (Renton, 1991)	1.207	1.275	1.703	5.582
$1/k$ (present)	1.207	1.275	1.720	5.753

5.2. Anisotropic material

In contrast to the isotropic case, a true validation of the present results for the anisotropic case is problematic. There are only a few known means of validation for the refined theory applied to anisotropic beams. One way is to compare with 3-D exact solutions. Unfortunately, this is probably not possible except in very simple cases and, even then, such solutions are unknown to the authors. Other ways include the generation of 3-D finite element results to a very high degree of accuracy, with perhaps millions of degrees of freedom, and carefully conceived experiments which somehow accentuate the role of the small terms. Such validation studies are beyond the scope of this paper and should be the topic of future research. About as far as one can go is to compare with results from other approaches, keeping in mind the asymptotic nature of the present formulation.

In particular, published results from *NABSA* (Nonhomogeneous Anisotropic Beam Section Analysis) can be used for comparison. A case considered by Hodges et al. (1992) is the strip-beam first analyzed by Minguet (1989). The cross section has width 1.182 in and thickness 0.05792 in and with the layup $[45°/0°]_{3s}$, which will be called the L1 configuration. The material is AS4/3501-6, which is the same in all of the following cases with the properties given in Table 7. The stiffness constants are given in Table 8. Unless otherwise specified, the order in which the stiffnesses are given is 1 — extension, 2,3 — shear, 4 — torsion, 5,6 — bending. Results from the 6×6 formulations bear the names *VABS* (for the present results) and *NABSA*, while results from the reduced 4×4 formulations are given under the names $NABSA_R$ and $VABS_R$. The reduced theory is developed by applying static condensation to the transverse shear rows and columns of the 6×6 matrix. In all cases the values in the reduced formulation of *VABS* have been found essentially identical to those of the classical formulation, as they must be. The classical formulation is very well validated, so this is a useful means to check the formulation; see Cesnik and Hodges (1997). The results obtained by applying the 'alternative theory' of Cesnik (1994) are given under the name $VABS_{AT}$.

The results from *NABSA* are pretty close to those obtained using *VABS*, except for the shear stiffness coefficient S_{33}. The *VABS* result for this coefficient, however, is closer to the one given by the alternative theory. The present results differ from both other theories in the case of the bending coefficient S_{55} by approximately 20%. It is noted, however, that the present result for S_{55} from $VABS_R$ is the same as that of the *VABS* classical formulation, a well-validated and asymptotically correct formulation.

Table 7
Material properties

E_{11}	20.6×10^6 psi
$E_{22} = E_{33}$	1.42×10^6 psi
$G_{12} = G_{13}$	0.87×10^6 psi
G_{23}	0.696×10^6 psi
$\nu_{12} = \nu_{13}$	0.3
ν_{23}	0.34

Table 8
Stiffness coefficients for L1 configuration

S	$NABSA$	$VABS$	$VABS_{AT}$	$NABSA_R$	$VABS_R$
S_{11}	0.8115×10^6	0.8112×10^6	0.8116×10^6	0.7884×10^6	0.7883×10^6
S_{12}	-0.4655×10^5	-0.4609×10^5	-0.4685×10^5	—	—
S_{22}	0.9368×10^5	0.9295×10^5	0.9291×10^5	—	—
S_{33}	0.6882×10^4	0.4034×10^3	0.4744×10^3	—	—
S_{44}	0.1251×10^3	0.1225×10^3	0.1290×10^3	0.1251×10^3	0.1194×10^3
S_{45}	0.3455×10^2	0.3030×10^2	0.3653×10^2	0.3455×10^2	0.2938×10^2
S_{55}	0.1852×10^3	0.2277×10^3	0.1864×10^3	0.1852×10^3	0.2275×10^3
S_{66}	0.9178×10^5	0.9177×10^5	0.9179×10^5	0.9178×10^5	0.9177×10^5

Other interesting cases are provided by box-beam configurations as shown in Fig. 3. The box beam under consideration has an exterior width of 0.953 in and a depth of 0.53 in with walls of total thickness 0.030 in. An antisymmetric box-beam configuration (B1) with a layup of $[15°]_6$ was also considered by Hodges et al. (1992). The results are given in Table 9. The results of both reduced theories show the significant reduction in bending stiffness due to the presence of bending–shear coupling, as the discussion of Rehfield et al. (1990) indicated should be present.

More cases of box beams are given by Bauchau and Hodges (1999). They have the same dimensions and material properties as the previous box beam, except the Poisson ratio is $\nu_{13} = 0.42$. The layups considered are given in Table 10. The order in which they are considered as indicated by the thick arrows in Fig. 3. The results for each layup are given in Tables 11–13. It can be seen in Table 11 that a very good agreement with the results from $NABSA$ is obtained in the absence of structural couplings. In Table 12, for an antisymmetric configuration that exhibits extension–twist and bending–shear couplings

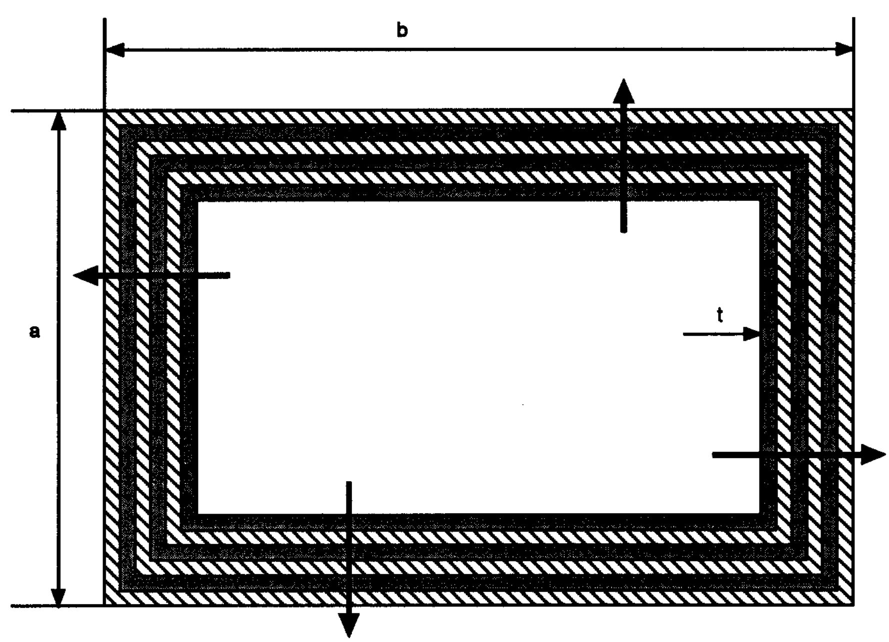

Fig. 3. Box-beam geometry.

Table 9
Stiffness coefficients for box-beam configuration B1

S	$NABSA$	$VABS$	$NABSA_R$	$VABS_R$
S_{11}	0.1438×10^7	0.1445×10^7	0.1438×10^7	0.1445×10^7
S_{14}	0.1075×10^6	0.1090×10^6	0.1075×10^6	0.1089×10^6
S_{22}	0.9018×10^5	0.5038×10^5	—	—
S_{25}	-0.5204×10^4	-0.2949×10^5	—	—
S_{33}	0.3932×10^5	0.2099×10^5	—	—
S_{36}	-0.5637×10^5	-0.2984×10^5	—	—
S_{44}	0.1678×10^5	0.1719×10^5	0.1678×10^5	0.1719×10^5
S_{55}	0.6622×10^5	0.5462×10^5	0.3619×10^5	0.3736×10^5
S_{66}	0.1726×10^6	0.1352×10^6	0.9179×10^5	0.9279×10^5

Table 10
Box-beam layups

Layup type	Upper wall	Lower wall	Left wall	Right wall
Layup 1	$[0°]_6$	$[0°]_6$	$[0°]_6$	$[0°]_6$
Layup 2	$[(30°,0°)_3]$	$[(30°,0°)_3]$	$[(30°,0°)_3]$	$[(30°,0°)_3]$
Layup 3	$[15°]_6$	$[-15°]_6$	$[\pm 15°]_6$	$[\pm 15°]_6$

Table 11
Stiffness coefficients for the box-beam configuration — layup 1

S	$NABSA$	$VABS$	$VABS_R$
S_{11}	0.177×10^7	0.177×10^7	0.177×10^7
S_{22}	0.444×10^5	0.445×10^5	—
S_{33}	0.190×10^5	0.191×10^5	—
S_{44}	0.816×10^4	0.819×10^4	0.819×10^5
S_{55}	0.869×10^5	0.869×10^5	0.869×10^5
S_{66}	0.215×10^6	0.215×10^6	0.215×10^3

the values follow the same trend. However, differences of up to 10% appear, for instance, in the shear stiffness S_{22}. For the symmetric configuration given by layup 3 which exhibits extension–shear, shear–shear, bending–twist and bending–bending couplings, the agreement becomes worse for the stiffness coefficients S_{13}, S_{23}, S_{33}, S_{46} and S_{56}. Note, however, that for the last two coefficients only the signs are different. Also, poor agreement with shear-related coefficients is seen in Table 9 where bending–shear coupling is present. While the present results are based on aymptotic methodology, the analysis behind *NABSA*, Giavotto et al. (1983), is not claimed to be asymptotic in any sense. Thus, because of the absence of results in the literature which would truly validate a refined theory of this type, no definitive conclusions can yet be drawn.

The last example for the transverse shear effect is provided by a comparison with the results in Rehfield et al. (1990) for the circular tube having a circumferentially uniform stiffness (CUS) layup

Table 12
Stiffness coefficients for the box-beam configuration — layup 2

S	$NABSA$	$VABS$	$VABS_R$
S_{11}	0.125×10^7	0.125×10^7	0.125×10^7
S_{14}	0.521×10^5	0.521×10^5	0.521×10^5
S_{22}	0.981×10^5	0.871×10^5	—
S_{25}	-0.264×10^5	-0.234×10^5	—
S_{33}	0.424×10^5	0.373×10^5	—
S_{36}	-0.278×10^5	-0.244×10^5	—
S_{44}	0.177×10^5	0.177×10^5	0.177×10^5
S_{55}	0.614×10^5	0.606×10^5	0.543×10^5
S_{66}	0.152×10^7	0.150×10^7	0.134×10^7

made of IM6/R6376 Graphite/Epoxy. The material properties are $E_{11}=23.1 \times 10^6$ psi, $E_{22}=1.4 \times 10^6$ psi, $\nu_{12}=0.338$, $G_{12}=0.73 \times 10^6$ psi. The layup is given by $[20,-70,20,(-70)_2,20]_T$. The elastic constants of the beam are given in Table 14. Differences appear for the shear stiffnesses and the shear–bending couplings. It must be mentioned that the theory of Rehfield et al. (1990) makes use of some quite restrictive assumptions about the transverse shear which are adopted from the theory for thin-walled beams made of isotropic material. These assumptions are that the shear strain is assumed constant along the entire circumference and that there is no warping due to transverse shear. None of these assumptions was made in the present approach.

To show the effects of the Timoshenko-like corrections relative to the classical theory for a composite beam, one must look at the effect of the beam length. In Fig. 4, 1-D results are compared for the stiffnesses provided by *NABSA* and *VABS* for box-beam configuration B1 (see above), but varying length. For a given cross section (so that b is fixed), the horizontal tip deflection $V=u_2(L)$ plotted as a function of L/b. The subscripts CBT and TBT refer to classical beam theory (i.e., one which uses the reduced 4×4 cross-sectional stiffness matrix from either VABS or NABSA) and Timoshenko-like beam theories based on stiffness coefficients from VABS and NABSA, respectively. One sees that as the beam becomes longer (i.e., slenderer) the Timoshenko-like theories based on stiffnesses from *VABS* and *NABSA* both converge to the classical result. As the beam becomes shorter, although the results are

Table 13
Stiffness coefficients for the box-beam configuration — layup 3

S	$NABSA$	$VABS$	$VABS_R$
S_{11}	0.137×10^7	0.137×10^7	0.992×10^6
S_{12}	-0.184×10^6	-0.184×10^6	—
S_{13}	0.144×10^3	0.176×10^4	—
S_{22}	0.884×10^5	0.883×10^5	—
S_{23}	-0.821×10^2	-0.842×10^3	—
S_{33}	0.395×10^5	0.775×10^4	—
S_{44}	0.173×10^5	0.174×10^5	0.174×10^5
S_{45}	0.180×10^5	0.180×10^5	0.180×10^5
S_{46}	0.358×10^3	-0.362×10^3	-0.362×10^3
S_{55}	0.608×10^5	0.608×10^5	0.608×10^5
S_{56}	0.377×10^3	-0.372×10^3	-0.372×10^3
S_{66}	0.143×10^6	0.143×10^6	0.143×10^6

Table 14
Stiffness coefficients for circular tube configuration — CUS

S	Rehfield et al. (1990)	*VABS*	$VABS_R$
S_{11}	0.1972×10^7	0.1886×10^7	0.1886×10^7
S_{14}	0.6680×10^6	0.6086×10^6	0.6086×10^6
S_{22}	0.2317×10^6	0.1137×10^6	—
S_{25}	-0.3340×10^6	-0.1609×10^6	—
S_{33}	0.2317×10^6	0.1137×10^6	—
S_{36}	-0.3340×10^6	-0.1609×10^6	—
S_{44}	0.4634×10^6	0.4159×10^6	0.4159×10^6
S_{55}	0.9862×10^6	0.7109×10^6	0.4831×10^6
S_{66}	0.9862×10^6	0.7109×10^6	0.4831×10^6

qualitatively identical, there is a noticeable quantitative difference in the behavior. The results using *VABS* properties are softer (yielding larger displacement) than those from *NABSA* properties. (It is noted, however, that in the few cases observed where reduced results from NABSA and VABS are not in agreement for one of the bending stiffnesses, the two sets of results would not converge to the same result in the limit of L/b tending to infinity).

6. Conclusions

A method capable of capturing the shear effects in beams made of arbitrary anisotropic material and of general cross-sectional shape has been developed. The method is based on asymptotic expansion of the energy in terms of a small parameter and then seeking the solution in a variational-asymptotic manner. Transverse shear effects are captured and a physical interpretation of the resulting of the shear variable was given.

It can be shown that for the situations when the stiffness matrix is diagonal, the problem has an exact solution. However, when off diagonal terms are present, the set of equations which must be solved is over-determined, so that the numerical results presented for such cases herein are determined from a minimization process. Although results cannot be said to be asymptotically exact for cases that have

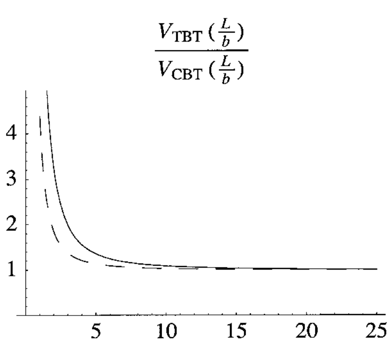

Fig. 4. Effect of beam slenderness on horizontal tip deflection due to horizontal tip shear load (specialized to linear theory); solid line — *VABS*; dashed line — *NABSA*.

non-diagonal cross-sectional stiffness matrices, the results presented are the best that can be obtained for a Timoshenko-like theory in the present asymptotically exact formulation. It is possible that different ways of posing the problem asymptotically could get around this problem or lead to very slightly improved results.

Results obtained for beams made of isotropic materials show very good agreement with energy-type approaches in the literature. The method predicts a variation of the shear stiffness coefficients with the breadth-to-depth ratio in the case of rectangular cross sections. Also, for a rectangular cross section there is a slight variation of the shear coefficient with Poisson's ratio. Cases with asymmetric cross-sectional geometries, or symmetric cases with the reference axes not aligned with either axis of symmetry, were studied for different Poisson's ratio. Here, off-diagonal shear couplings appear and a set of principal axes associated with transverse shear can be determined which are, in general, different from the principal axes of bending.

Beams made of composite material are more challenging due to the presence of different couplings and also because few results are available in the literature to allow comparison. Results available for strip-like configurations show good agreement with previous approaches. As expected, relatively large changes in the bending stiffnesses were noticed for certain layups especially when bending–shear couplings are present. For box beams the level of agreement with previous results varies from excellent in the case of diagonal stiffness matrices to cases when relatively large variations of some of the shear rigidity terms were found for certain layups. The stiffness model developed herein is somewhat softer than that developed by Giavotto et al. (1983), which might imply that the latter model is somehow overconstrained. However, the ultimate validation of a model to capture the transverse shear effect for composite beams is beyond the scope of this paper. Indeed, comprehensive validation of the refined model would require exact 3-D solutions, highly refined 3-D finite element computations, or possible experiments.

The most important contribution of the present work concerning composite beams is the implied proof that, since the problem for the general case is overdetermined, "an asymptotically correct Timoshenko-like theory cannot be obtained in the general case". However, an approximate solution of the resulting overdetermined set of equations can be found in terms of a least-squares minimization, which still guarantees the best results within the given approximation. Even though entirely different approaches were used, the present determinations are parallel to the conclusions of Sutyrin and Hodges (1996) for composite plates. A fundamental difference between the plate and the beam problems, however, is that for plates the problem solved for the elastic constants is 1-D whereas for beams it is 2-D, considerably more complex. From another perspective, plate analysis reduces the 3-D elasticity formulation to a 2-D model, while for beams the 3-D formulation undergoes a more serious reduction to a 1-D model. This is why a closed-form analytical solution for dimensional reduction of composite beams is so much more involved, in general.

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